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Novel results for the stability of h-discrete fractional neural networks with nonsingular and nonlocal kernels

Research Article

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Abstract: We investigate a novel type of nonlinear discrete-time fractional neural networks using the h-discrete nabla ABC fractional operator. To derive appropriate criteria for the existence and uniqueness of solutions to the issues at hand, we apply basic fixed-point theory techniques. Furthermore, The Ulam-Hyers stability of the considered model, as well as significant findings, are shown. Moreover, to highlight the validity of the presented conclusions, two and three-dimensional examples are explored.

MSC: 39A12, 39B82

Keywords: discrete-time fractional neural networks • h-discrete nabla ABC fractional operator • Schauder fixed point theorem • Ulam-Hyers stability

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1. Introduction

Fractional calculus is a branch of applied mathematics that works with derivatives and integrals of arbitrary orders, and its applications may be found in science, engineering, applied mathematics, economics, and other areas [1][2][3][4][5][6]. Knowing that fractional calculus in general has an extensive and deep history, discrete fractional calculus in particular is yet introduced as a new promising field of research that has attracted the interest of many researchers. Similar to the theory of fractional calculus, the theory of discrete fractional calculus has progressed in numerous directions over this time period (see [7][8][9][10][11]).
Neural networks have received a lot of attention during the last few decades. This is mostly owing to its wide range of applications in fields such as pattern recognition, associative memory and model identification [12][13][14]. As

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is well known, these applications mainly rely on the dynamical features of neural networks, hence, incorporating fractional calculus into neural networks gives a more precise tool for describing memory and heredity properties of various processes, as well as improving the dynamical system's design, description, and control capabilities. Take for example, [15][16][17][18][19]. In fact, discrete-time fractional-order neural networks have been widely employed in image processing [20] and time series analysis [21]. Because these fractional discrete neural networks have exact discrete relations, we may obtain numerical simulation. As an example, one of the most significant dynamics that is investigated and addressed is stability analysis [22][23][24].

From among mathematical models described in discrete fractional calculus, discrete AB-fractional operators, which were used to build current operators and their characterizations, were suggested in a research study [25][26]. Furthermore, discrete fractional calculs has been conceptually presented further by proposing and analyzing discrete versions of these fractional operators [27].

The classic Ulam-Hyers stability was uncovered in a functional equations lecture and has progressively been known by academics, with more and more scholars beginning to research the Ulam-Hyers stability of various forms of fractional-order equations [28] [29][30]. In discrete-time equations and systems, the Ulam-Hyers stability has also been steadily evolved. The Ulam-Hyers stability of a family of discrete fractional equations with anti-periodic boundary conditions was discussed in [31]. In [32] the stability of nonlinear discrete fractional initial value problems with application to vibrating eardrum in the sens of Ulam-Hyers stability was investigated, [33] provides Ulam–Hyers stability results for Caputo nabla fractional difference equations in both linear and nonlinear cases, while [34] demonstrated the existence and Ulam-Hyers stability of solutions for an initial value discrete fractional Duffing equation with a forcing term. However, there are few results on the Ulam-Hyers stability of discrete-time fractional-order neural networks we state [35], that investigated the Hyers-Ulam stability of a linear fractional neural network. As a result, Ulam-Hyers stability of fractional-order neural networks on a discrete-time scale has promising research potential.

The overall aim of this study is to provide significant stability results for discrete Fractional Neural Networks with h-discrete nabla ABC operator. We will go over the ABC h-discrete nabla fractional neural network. then, We present criteria for the presence of a solution to such a discrete-time system, then we address the Ulam-Hyers stability of the proposed neural network and derive essential conclusions.

Based on the preceding discussion, the paper is set as follows: Section 2 contains an introduction to discrete fractional calculus, as well as several definitions and important properties. Section 3 consists of the presentation of the fractional-order discrete-time neural network based on the Caputo AB nabla discret difference operator and a crucial theorem addressing the existence of the solution. The Ulam-Hyers stability is adressed in Section 4, important lemma and theorem are concluded regarding this stabiilty. Section 5 presents two numerical examples with simulations to demonstrate the validity and relevance of the theoretical outcomes.

2. Mathematical background

The following definitions for the discrete fractional calculus are introduced. We use the notation $\mathbb{N}_{a,h} = \{a, a+h, a+2h, ...\}$

Definition 2.1 ([36]).
The backward difference operator on $h\mathbb{Z}$ is defined as

$$\nabla_h u(t) = \frac{u(t) - u(t-h)}{h}, \tag{1}$$

the increasing h-polinomial factorial function is defined as

$$t_h^{\overline{\alpha}} = h^\alpha \frac{\Gamma(\frac{t}{h} + \alpha)}{\Gamma(\frac{t}{h})}, \quad t, \alpha \in \mathbb{R} \tag{2}$$

where Γ is the Gamma function

Definition 2.2 ([36]).
Let u be defined on $\mathbb{N}_{a,h} \cup {}_{a,h}\mathbb{N}$, $a < b$, $a = b$ (mod h) $v \in [0,1]$ such that $|\lambda h^v| < 1$, then the left nabla ABC fractional difference (in the sense of Atangana and Baleanu) is defined by:

$${}^{ABC}_{a}\nabla^v_h u(t) = H(v,h)\frac{1-v+vh}{1-v} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} h\nabla_h u(sh)\, {}_hE_{\overline{v}}(\lambda, t-\rho(sh)), \tag{3}$$

where

$$H(v,h) = B(v)[\frac{v}{h} + (1-v)], \tag{4}$$

and

$$B(v) = 1 - v + \frac{v}{\Gamma(v)}, \tag{5}$$

${}_hE_{\overline{v}}$ is the Nabla h-discrete Mittag-Leffler function described by

$${}_hE_{\overline{v,;\xi}}(\lambda, w) = \sum_{k=0}^{\infty} \lambda^k \frac{w_h^{\overline{kv+\xi-1}}}{\Gamma(vk+\xi)}, \tag{6}$$

Definition 2.3 ([36]).
The left h-fractional sum associate to ${}^{ABC}_{a}\nabla^v_h u(t)$ with order $0 < v < 1$ is defined on $\mathbb{N}_{a,h}$ by

$${}^{AB}_{a}\nabla_h^{-v} u(t) = \frac{1-v}{H(v,h)(1-v+vh)} u(t) + \frac{v}{H(v,h)(1-v+vh)} \left({}_a\nabla_h^{-v} u\right)(t), \tag{7}$$

Definition 2.4 ([36]).
Let $x : \mathbb{N}_{a,h} \to \mathbb{R}$ and $0 < \alpha$ be given. a is a starting point. The $v-$th order h-sum is given by

$${}_a\nabla_h^{-\alpha} x(t) = \frac{h}{\Gamma(\alpha)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t - \rho(sh))_h^{\overline{\alpha-1}} x(sh), \quad \rho(sh) = (s-1)h, \quad t \in \mathbb{N}_{a,h} \tag{8}$$

Lemma 2.1 ([36]).
For $v > 0$ and $\gamma > -1$ the following holds

$${}_a\nabla_h^{-v} (t-a)^{\overline{\gamma}} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+v)} (t-a)_h^{\overline{v+\gamma}}, \tag{9}$$

3. Existence of the solution

We propose the following fractional-order discrete-time neural network

$$ {}_{a}^{ABC}\nabla_{h}^{v}x(t) = -Ax(t) + Bf(t, x(t)) + I, \tag{10} $$

Where ${}_{a}^{ABC}\nabla_{t}^{\alpha(t)}$ is the Caputo AB nabla discret difference operator with order $\alpha(t)$, $0 < \alpha(t) < 1$, $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$ is the state vector, $A = diag(a_1, a_2, ..., a_n) \in \mathbb{R}^{n*n}$ is the self-feedback connection weight with $a_i > 0$, $B = (b_{ij})_{n*n} \in \mathbb{R}^{n*n}$ is the connection weight matrix, $g(x(t)) = (g_1(x(t)), g_2(x(t)), ..., g_n(x(t)))^T : C(\mathbb{N}_{a+1} \to \mathbb{R}^n)$ is the acctivation function, $I = (I_1, ..., I_n)^T$ the vector of external inputs.

In the remainer of our study we consider the external inputs vector to be $0_{\mathbb{R}^n}$. In order to conduct our research, we must propose the following two hypothesis

(H_1) The acctivation function is continous and verify the lypschitz condition

ie there exists a positive constant F_i such that

$$ |f_i(t, u) - f_i(t, v)| \leq |u - v|, \quad u, v \in \mathbb{R} \tag{11} $$

(H_2) There exists a constant $k > 0$ such that

$$ k = \frac{\gamma_1 + F\gamma_2}{H(v,h)(1 - v + vh)}(1 - v + \frac{vh}{\Gamma(v+1)}(T-a)_h^{\overline{v}}) < 1, \tag{12} $$

where

$$ \gamma_1 = \max_{i=1,\dots,n} a_i, \quad \gamma_2 = \max_{i=1,\dots,n} \sum_{j=1}^{n} |b_{ij}| \quad \text{and} \quad F = \max_{i=1,\dots,n} F_i $$

Our first existence result is based on Schauder's fixed point theorem. We demonstrate that the operator Ψ defined by (13) meets the hypothesis of the fixed point theorem of Schauder.

where the operator Ψ is considered as

$$ \Psi_i x_i(t) = x_i(a) + \frac{1 - v}{H(v,h)(1 - v + vh)}[-a_i x_i(t) + \sum_{i=1}^{n} b_{ij} f_j(t, x_j(t)) + I_i] \tag{13} $$

$$ + \frac{v}{H(v,h)(1 - v + vh)} {}_{a}\nabla_{h}^{-v} \left(-a_i x_i(t) + \sum_{i=1}^{n} b_{ij} f_j(t, x_j(t)) + I_i \right) \tag{14} $$

We define $S = \{x \in C(\mathbb{N}_{a,h}^{T}; \mathbb{R}^n), \|x\| \leq \kappa\}$, where $C(\mathbb{N}_{a,h}^{T}; \mathbb{R}^n)$ denotes the set of continuous functions from $\mathbb{N}_{a,h}^{T}$ to $\mathbb{R}^n$ and $\mathbb{N}_{a,h}^{T} = \{a, a+h, a+2h, ..., T\}$ with a bounded initial condition $\|x_a\| \leq \xi$

Theorem 3.1.
If (H_1) and (H_2) are valid and if the following inquality is true

$$\kappa \geq \frac{\xi}{1 - \frac{\gamma_1 + F\gamma_2}{H(v,h)(1-v+vh)}(1 - v + \frac{vh}{\Gamma(v+1)}(T-a)_h^{\overline{v}})}, \tag{15}$$

Then, the discrete fractional-order neural network (10) has at least one solution.

Proof. Clearly, S is a nonempty, closed, bounded and convexe subset of $\mathbb{R}^n$.

First, we proove that Ψ is continuous. Consider a sequence $\{y_n\} \subset S$ such that $y_n \to y$ in S. To show that Ψ is continuous, we have to prove that

$$\|\Psi y_n - \Psi y\| \to 0, \quad \text{when} \quad n \to \infty$$

Using (H_1) we have

$$\begin{aligned}
|\Psi_i y_{ni}(t) - \Psi_i y_i(t)| = |&\frac{1-v}{H(v,h)(1-v+vh)}[-a_i(y_{ni}(t) - y_i(t)) + \sum_{j=1}^{n} b_{ij}(f_j(t, y_{ni}(t)) - f_j(t, y_j(t)))] \\
&+ \frac{vh}{H(v,h)(1-v+vh)\Gamma(v)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t - \rho(sh))_h^{\overline{v-1}}[-a_i(y_{ni}(sh) - y_i(sh)) \\
&+ \sum_{i=1}^{n} b_{ij}(f_j(sh, y_{nj}(sh)) - f_j(sh, y_j(sh)))]| \\
&\leq \frac{1-v}{H(v,h)(1-v+vh)}[a_i + F_i \sum_{j=1}^{n} |b_{ji}|]|y_{ni}(t) - y_i(t)| \\
&+ \frac{vh}{H(v,h)(1-v+vh)\Gamma(v)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t - \rho(sh))_h^{\overline{v-1}}[a_i|y_{ni}(sh) - y_i(sh)| \\
&+ F_i \sum_{i=1}^{n} |b_{ji}||y_{nj}(sh)) - y_j(sh))|]
\end{aligned}$$

Which lead us to with the help of (H_2)

$$\begin{aligned}
\|\Psi y_n(t) - \Psi y(t)\| \leq &\frac{1-v}{H(v,h)(1-v+vh)}[\gamma_1 + F\gamma_2]\|y_n(t) - y(t)\| \\
&+ \frac{vh}{H(v,h)(1-v+vh)\Gamma(v)}[\gamma_1 + F\gamma_2]\|y_n(t) - y(t)\| \sup \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t - \rho(sh))_h^{\overline{v-1}} \\
&= \frac{\gamma_1 + F\gamma_2}{H(v,h)(1-v+vh)}(1 - v + \frac{vh}{\Gamma(v+1)}(T-a)_h^{\overline{v}})\|y_n(t) - y(t)\|
\end{aligned}$$

Since

$$\|\Psi y_n - \Psi y\| \to 0, \quad \text{for} \quad y_n \to y,$$

Consequently, Ψ is continuous.

Then, we show that Φ maps bounded sets into bounded sets ie $\Phi(S)\subset S$. We have that for each $t\in\mathbb{N}_{a,h}^{T}$

$$\begin{aligned}|\Psi_i y_i(t)| &\leq |y_i(a)| + \frac{1-v}{H(v,h)(1-v+vh)}[a_i|y_i(t)| + \sum_{i=1}^{n}|b_{ij}|F_j|y_j(t)| + |I_i|] \\ &+ \frac{v}{H(v,h)(1-v+vh)}\,{}_a\nabla_h^{-v}\left(a_i|y_i(t)| + \sum_{i=1}^{n}|b_{ij}|F_j|y_j(t)| + |I_i|\right) \\ &= |y_i(a)| + \frac{1-v}{H(v,h)(1-v+vh)}[(a_i + F_i\sum_{i=1}^{n}|b_{ji}|)|y_i(t)| + |I_i|] \\ &+ \frac{v}{H(v,h)(1-v+vh)}\,{}_a\nabla_h^{-v}\left((a_i + F_i\sum_{i=1}^{n}|b_{ij}|)|y_i(t)| + |I_i|\right)\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|\Psi y(t)\| &\leq \xi + \frac{1-v}{H(v,h)(1-v+vh)}[(\gamma_1 + F\gamma_2)\kappa + \gamma_3] \\ &+ \frac{v(T-a)_h^{\overline{v}}}{H(v,h)(1-v+vh)\Gamma(v+1)}[(\gamma_1 + F\gamma_2)\kappa + \gamma_3] \\ &= \xi + \frac{\gamma_1 + F\gamma_2}{H(v,h)(1-v+vh)}(1-v+\frac{v(T-a)_h^{\overline{v}}}{\Gamma(v+1)})\kappa + \frac{\gamma_3}{H(v,h)(1-v+vh)}(1-v+\frac{v(T-a)_h^{\overline{v}}}{\Gamma(v+1)}) \\ &\leq \kappa\end{aligned}$$

Since, $\|\Psi y(t)\|\leq\kappa$ then, maps bounded sets into bounded sets.

Now, we prove the equicontinuity of the operator Ψ_i. For this, we consider

$$\begin{aligned}|\Psi_i y_i(t_1) - \Psi_i y_i(t_2)| &\leq \frac{1-v}{H(v,h)(1-v+vh)}[a_i|y_i(t_1) - y_i(t_2)| + \sum_{i=1}^{n}|b_{ij}|F_j|y_j(t_1) - y_j(t_22)|] \\ &+ \frac{v}{H(v,h)(1-v+vh)}\,{}_a\nabla_h^{-v}\left(a_i|y_i(t_1) - y_i(t_2)| + \sum_{i=1}^{n}|b_{ij}|F_j|y_j(t_1) - y_j(t_2)|\right) \\ &= \frac{1-v}{H(v,h)(1-v+vh)}(a_i + F_i\sum_{i=1}^{n}|b_{ji}|)|y_i(t_1) - y_i(t_2)| \\ &+ \frac{v}{H(v,h)(1-v+vh)}(a_i + F_i\sum_{i=1}^{n}|b_{ij}|)\,{}_a\nabla_h^{-v}(|y_i(t_1) - y_i(t_2)|)\end{aligned}$$

which leads us to

$$\begin{aligned}\|\Psi y(t_1) - \Psi y(t_2)\| &\leq \frac{1-v}{H(v,h)(1-v+vh)}(\gamma_1 + F\gamma_2)\|y(t_1) - y(t_2)\| \\ &+ \frac{v}{H(v,h)(1-v+vh)}(\gamma_1 + F\gamma_3)\,{}_a\nabla_h^{-v}(\|y(t_1) - y(t_2)\|) \\ &\leq \frac{\gamma_1 + F\gamma_2}{H(v,h)(1-v+vh)}(1-v+\frac{v}{\Gamma(v+1)}(T-a)^{\overline{v}_h})\|y(t_1) - y(t_2)\|\end{aligned}$$

We infer that Ψ is an equicontinuous set since $\|\Psi y(t_1) - \Psi y(t_2)\|\to 0$ when $t_1\to t_2$. Because $\Psi(S)\subset S$, ψ is obviously uniformly bounded. Ψ is a compact operator according to the Arzelá–Ascoli theorem. As a result of the Schauder fixed point theorem, the operator Ψ has a fixed point, indicating that problem (10) has a solution.

□

4. Ulam-Hyers stability

Definition 4.1.
The discrete fractional initial value problem (10) is Hyers–Ulam stable if there exists $U > 0$ such that for any $\epsilon > 0$, satisfies

$$\| {}_{a}^{ABC}\nabla_h^v x(t) + Ax(t) - Bf(t, x(t)) - I\| \leq \epsilon, \tag{16}$$

Then there is a solution $y(t)$ of (10) such that

$$\|x(t) - y(t)\| \leq U\epsilon, \tag{17}$$

Lemma 4.1.
If x solves (10), then,

$$\begin{aligned}\|x(t) - x_a - &\frac{1-v}{H(v,h)(1-v+vh)}[-Ax(t) + Bf(t, x(t)) + I] \\ &- \frac{vh}{H(v,h)(1-v+vh)\Gamma(v)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{v-1}} (-Ax(sh) + Bf(sh, x(sh)) + I) \| \leq \kappa\epsilon\end{aligned}$$

where

$$\kappa = \frac{1}{H(v,h)(1-v+vh)}(1 - v + \frac{v}{\Gamma(v+1)}(T-a)_h^{\overline{v}})) \tag{18}$$

Proof. A function $x(t)$ solves (10) if and only if it exists $g(t)$ satisfing

$${}_{a}^{ABC}\nabla_h^v x(t) + Ax(t) - Bf(t, x(t)) - I = g(t), \tag{19}$$

and

$$\|g(t)\| \leq \epsilon, \tag{20}$$

Therefore, we have

$$\begin{aligned}|x_i(t) - x_i(a) - &\frac{1-v}{H(v,h)(1-v+vh)}[-a_i x_i(t) + \sum_{i=1}^{n} b_{ij} f_j(t, x_j(t)) + I_i] \\ &- \frac{vh}{H(v,h)(1-v+vh)\Gamma(v)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{v-1}} \left(-a_i x_i(sh) + \sum_{i=1}^{n} b_{ij} f_j(sh, x_j(sh)) + I_i\right) | \\ &= |\frac{1-v}{H(v,h)(1-v+vh)} g_i(t) + \frac{vh}{H(v,h)(1-v+vh)\Gamma(v)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{v-1}} g_i(sh)| \\ &\leq \frac{1-v}{H(v,h)(1-v+vh)} |g_i(t)| + \frac{vh}{H(v,h)(1-v+vh)\Gamma(v)} \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} (t-\rho(sh))_h^{\overline{v-1}} |g_i(sh)|\end{aligned}$$

We obtain

$$
\begin{aligned}
\|x(t)-x_a-&\frac{1-v}{H(v,h)(1-v+vh)}[-Ax(t)+Bf(t,x(t))+I]\\
&-\frac{vh}{H(v,h)(1-v+vh)\Gamma(v)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{v-1}}(-Ax(sh)+Bf(sh,x(sh))+I)\|\\
&\leq\frac{1}{H(v,h)(1-v+vh)}(1-v+\frac{v}{\Gamma(v+1)}(T-a)_h^{\overline{v}}))\|g(t)\|\\
&\leq\kappa\epsilon
\end{aligned}
$$

Which completes the proof. □

Theorem 4.1.
Under hypothesis (H_1) and (H_2) the ABC h-discrete fractional-order neural network (10) is Ulam-Hyers stable.

Proof. . Let $\epsilon>0$ and let $x\in C(\mathbb{N}_{a,h},\mathbb{R}^n)$ be a function which satisfies Lemma 4.1 and let $x\in C(\mathbb{N}_{a,h},\mathbb{R}^n)$ be the unique solution of (10). For each $t\in\mathbb{N}_{a,h}$, we have

$$
\begin{aligned}
|x_i(t)-y_i(t)|=&|x_i(t)-x_a-\frac{1-v}{H(v,h)(1-v+vh)}[-a_iy_i(t)+\sum_{i=1}^{n}b_{ij}f_j(t,y_j(t))+I_i]\\
&-\frac{vh}{H(v,h)(1-v+vh)\Gamma(v)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{v-1}}\left(-a_iy_i(t)+\sum_{i=1}^{n}b_{ij}f_j(t,y_j(t))+I_i\right)\\
&+\frac{1-v}{H(v,h)(1-v+vh)}[-a_ix_i(t)+\sum_{i=1}^{n}b_{ij}f_j(t,x_j(t))+I_i]\\
&+\frac{vh}{H(v,h)(1-v+vh)\Gamma(v)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{v-1}}\left(-a_ix_i(t)+\sum_{i=1}^{n}b_{ij}f_j(t,x_j(t))+I_i\right)\\
&-\frac{1-v}{H(v,h)(1-v+vh)}[-a_ix_i(t)+\sum_{i=1}^{n}b_{ij}f_j(t,x_j(t))+I_i]\\
&-\frac{vh}{H(v,h)(1-v+vh)\Gamma(v)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{v-1}}\left(-a_ix_i(t)+\sum_{i=1}^{n}b_{ij}f_j(t,x_j(t))+I_i\right)\\
&\leq|x_i(t)-x_a-\frac{1-v}{H(v,h)(1-v+vh)}[-a_ix_i(t)+\sum_{i=1}^{n}b_{ij}f_j(t,x_j(t))+I_i]\\
&-\frac{vh}{H(v,h)(1-v+vh)\Gamma(v)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{v-1}}\left(-a_ix_i(t)+\sum_{i=1}^{n}b_{ij}f_j(t,x_j(t))+I_i\right)|\\
&+\frac{a_i+\sum_{i=1}^{n}|b_{ji}|F_i}{H(v,h)(1-v+vh)}[(1-v)|x_i(t))-y_i(t)|+v\frac{h}{\Gamma(v)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{v-1}}|x_i(t))-y_i(t)|]
\end{aligned}
$$

Then, we have

$$\begin{aligned}\|x(t)-y(t)\| &\leq \kappa\epsilon+\frac{\gamma_1+F\gamma_2}{H(v,h)(1-v+vh)}[(1-v)+v\sup\left\{\frac{h}{\Gamma(v)}\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}}(t-\rho(sh))_h^{\overline{v-1}}\right\}]\|x(t))-y(t)\| \\ &\leq \kappa\epsilon+\frac{\gamma_1+F\gamma_2}{H(v,h)(1-v+vh)}[(1-v)+\frac{v}{\Gamma(v+1)}(T-a)^{\overline{v}_h}\|x(t))-y(t)\| \\ &\leq \frac{\kappa}{1-k}\epsilon\end{aligned}$$

For $U=\dfrac{\kappa}{1-k}$, according to Definition 4.1 system (10) is Ulam-Hyers stable. □

5. Numerical simulations

In this section, we will come across two examples with numerical simulations to demonstrate the relevance and accuracy of our theoretical results.

Example 1 We consider the two dimensional discrete-time neural network

$$\begin{cases} {}_a^{ABC}\nabla_h^v x_1(t) &= -a_1x_1(t)+b_{11}\sin(x_1(t))+b_{12}sin(x_2(t))+I_1, \\ {}_a^{ABC}\nabla_h^v x_2(t) &= -a_2x_2(t)+b_{21}\sin(x_1(t))+b_{22}sin(x_2(t))+I_2; \end{cases} \tag{21}$$

With the following parameters

$$v=\frac{1}{3};\quad h=0.55;\quad a_1=0.25;\quad a_2=0.2;\quad b_{11}=-0.1;\quad b_{12}=0.05;\quad b_{21}=0.15;\quad b_{22}=-0.1.$$

and $\quad I_1=0;\;\; I_2=0;$

where the initial condition is $x(0)=(0.5,0.5)^T$

In this case, assumptions (H_1) and (H_2) are valid for $k=0.924652<1$. Therefore, according to Theorem 3.1, we conclude that system (21) has at least one solution. For $U=24.5435$, Theorem 4.1 is valid and (21) is Ulam-Hyers stable.

We provide the following numerical formula and Figure 1 to illustrate the results described above

$$\begin{cases} x_1(i)=x_1(0)+\dfrac{1-v}{H(v,h)(1-v+vh)}[-a_1x_1(i)+b_{11}\sin(x_1(i))+b_{12}\sin(x_2(i))+I_1] \\ \quad +\dfrac{vh^v}{H(v,h)(1-v+vh)\Gamma(v)}\sum_{j=1}^{i}\dfrac{\Gamma(i-j+v)}{\Gamma(i-j+1)}\left(-a_1x_1(j)+b_{11}\sin(x_1(j))+b_{12}\sin(x_2(j))+I_1\right), \\ x_2(i)=x_2(0)+\dfrac{1-v}{H(v,h)(1-v+vh)}[-a_2x_2(i)+b_{21}\sin(x_1(i))+b_{22}\sin(x_2(i))+I_2] \\ \quad +\dfrac{vh^v}{H(v,h)(1-v+vh)\Gamma(v)}\sum_{j=1}^{i}\dfrac{\Gamma(i-j+v)}{\Gamma(i-j+1)}\left(-a_2x_2(j)+b_{21}\sin(x_1(j))+b_{22}\sin(x_2(j))+I_2\right), \end{cases} \tag{22}$$

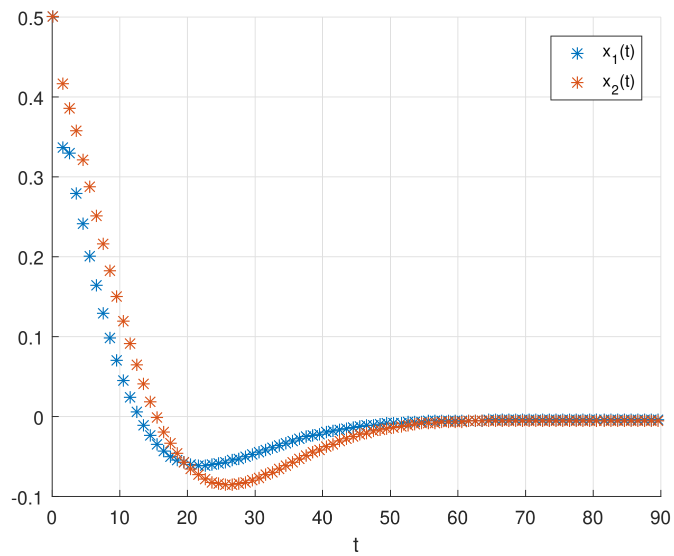

Figure 1. **Numerical solution of fractional-order discrete-time neural network (21)**

Example 2 Let be the followinng fractional discrete-time neural network

$$ {}_{a}^{ABC}\nabla_{h}^{v}x(t) = -Ax(t) + B\tanh(x(t)) + I, \tag{23} $$

where

$$ A = diag(0.4, 0.4, 0.4); \quad B = \begin{bmatrix} 0.01 & -0.05 & 0.03 \\ 0.01 & -0.06 & 0.01 \\ 0.02 & -0.04 & 0.03 \end{bmatrix}; \quad I = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

and

$$ \tanh(x(t)) = (\tanh(x_1(t)), \tanh(x_2(t)), \tanh(x_3(t)))^T; \quad h = 0.35; \quad v = 0.5 $$

with the initial condition $x(0) = (-0.2, 0.1, 0.2)^T$

the accuracy of (H_1) and (H_2) are obtained as $k = 0.9053$. On the other hand, $\kappa \approx 2.11193$ and $U = 19.509$ which satisfies both Theorem 3.1 and Theorem 4.1, therefore, it exists a solution of problem (10) which is Ulam-Hyers stable.

The numerical solution of the discrete neural network (23) is shown in Figure 2 with the help of the numerical formula (24)

$$ \begin{cases} x(i) = x(0) + \dfrac{1-v}{H(v,h)(1-v+vh)}[-Ax(i) + B\tanh(x(i)) + I] \\ \quad + \dfrac{vh^v}{H(v,h)(1-v+vh)\Gamma(v)} \sum_{j=1}^{i} \dfrac{\Gamma(i-j+v)}{\Gamma(i-j+1)} (-Ax(j) + B\tanh(x(j)) + I) \end{cases} \tag{24} $$

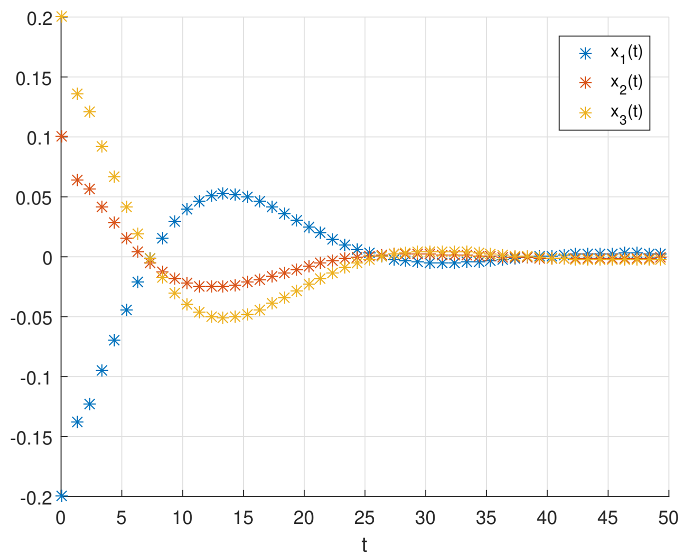

Figure 2. **Numerical solution of discrete neural network (23)**

6. Conclusion

This research contributes to the issue of the stability of fractional discrete neural networks by introducing a fractional-order network model based on the nabla h-discrete fractional operator with nonsingular and nonlocal kernels and establishing its Ulam-Hyers stability. Namely, two unique theorems were proven: one concerning the existence of the solution for the suggested fractional-order model, and the other addressing its Ulam-Hyers stability. Finally, numerical simulations were performed to demonstrate the efficiency of the theoretical method described herein.

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