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# A regularity in Ramanujan summation function results for the triangular number series

**Research Article**

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**Abstract:** The Triangular Number Series, defined as $1 + 2 + 3 + 4 + \cdots$, when subjected to Ramanujan Summation, gives the known and somewhat controversial result of $-\frac{1}{12}$. The Ramanujan Summation Function is defined in such a way as to accept any such series and produce corresponding values for them, thus allowing for the aforementioned result to be obtained as well. In this paper, the Triangular Number Group Series Function is defined as a function that generates members of a sum, whereas the values of those members depend on a parameter $g$, representing the number of elements from the Triangular Number Series in each member of the new sum. This paper shows that the results of the Ramanujan Summation Function upon any such Triangular Number Group Series follow a regularity that is also dependent on the parameter $g$. Once such a regularity is obtained, the scope of $g$ is extended to real and complex numbers as well.

**Keywords:** Ramanujan Summation Function • Triangular Number Series • Regularity

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## 1. Introduction

Ramanujan Summation is a technique for assigning values to divergent infinite series, for which traditional summation does not apply, since the partial sums generated by such series don't converge. Even though techniques for summation of divergent sums were known before Srinivasa Ramanujan's time, it was generally accepted that infinite divergent sums cannot be assigned proper values, even during Ramanujan's lifetime (e.g., [12]). However, a discovery of Ramanujan's "lost notebook" and a treatise of

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the results and ideas contained therein [1, 2, 3, 4, 5] sparked a resurgence in interests in Ramanujan's works, including Ramanujan Summation. Its relations to other techniques for divergent series summation can be seen in [7]. A thorough treatise of Ramanujan Summation is given in [6].

One of the consequences of using Ramanujan Summation is to give grounds to the claim that $1+2+3+4+\cdots = -\frac{1}{12}$. This claim has been arguably popularized the most by the YouTube channel Numberphile and the videos claiming to show proofs for the aforementioned claim. Two videos stand out, namely a video claiming to show a proof by arithmetic manipulation using infinity [8] and a video claiming to show a proof using the Riemann zeta function [9]. In the latter video it is stated that the aforementioned value of $-\frac{1}{12}$ is obtained as a result of analytic continuation and this seems to be the currently accepted stance towards it, namely that $1+2+3+4+\cdots$ is not *equal* to $-\frac{1}{12}$, but that it is *associated* with $-\frac{1}{12}$ [11, 13, 16].

If a function $f(n)$ has the following properties: 1) it is defined for $n = 1, 2, 3, 4, 5, \ldots$ ; 2) it is not infinitely derivable; and 3) it has no divergence at $n = 1$, then the Ramanujan Summation can be applied to it as follows: $f(1) + f(2) + f(3) + f(4) + \ldots = -\frac{f(0)}{2} + i \cdot \int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt$ . Thus, the *Ramanujan Summation Function*, given as the following Equation (1), can be defined for functions with the aforementioned properties.

$$R\big(f(n)\big) = -\frac{f(0)}{2} + i \cdot \int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt \tag{1}$$

Given the function $\mathrm{T}(n) = n$, the infinite series $T(1) + T(2) + T(3) + T(4) + \cdots = 1+2+3+4+\cdots$ can be obtained and this is the *Triangular Number Series*, because the partial sums of this series equal 1, 3, 6, 10, 15, …, i.e., they form the Triangular Number Sequence [10]. Since for the partial sums it equals that $\sum_{k=1}^{n} T(k) = \frac{n\cdot(n+1)}{2}$, standard infinite summation would require that the partial sums be calculated with the argument going towards infinity, which would yield infinity as the result. However, since $\mathrm{T}(n)$, as a function, has the properties that the Ramanujan Summation Function requires, the obtained result is $R\big(T(n)\big) = -\frac{1}{12}$.

If the Triangular Number Series were convergent, rearranging its terms would yield the same result, regardless of the rearrangement. However, since the series is divergent, it would be expected that the outcomes of rearranging its terms would be different, depending on the rearrangements. The question that was asked was: what would be the results of submitting various versions of the Triangular Number Series to the Ramanujan Summation Function, if the versions of the series followed some regularity, and also would a corresponding regularity be observed in the results of the Ramanujan Summation Function as well? In particular, the regularity that was attempted was to group the terms of the series as $(1+2) + (3+4) + (5+6) + \cdots = 3 + 7 + 11 + \cdots$. Another example was to group the terms of the series as $(1+2+3) + (4+5+6) + (7+8+9) + \cdots = 6 + 15 + 26 + \cdots$ etc. The motivation was to find out what would be the outcomes of applying the Ramanujan Summation Function upon such series.

## 2. The Triangular Number Group Series Function

Each Triangular Number Group Series will be denoted as $T$, where $g$ will denote the number of elements in a group. Thus, $T = T_1 = 1 + 2 + 3 + 4 + \cdots$; $T_2 = (1 + 2) + (3 + 4) + (5 + 6) + \cdots = 3 + 7 + 11 + \cdots$ ; $T_3 = (1 + 2 + 3) + (4 + 5 + 6) + (7 + 8 + 9) + \cdots = 6 + 15 + 24 + \cdots$ etc. Each individual member of such a series will be denoted as $T_g(n)$, where $n$ is the index of each member of the particular Group Series. Thus, $T_1(4) = 4$; $T_2(4) = (7 + 8) = 15$; $T_3(4) = (10 + 11 + 12) = 33$ etc. Therefore, $T_g(n)$ will be called the *Triangular Number Group Series Function,* i.e., the function using which each individual term of the corresponding Triangular Number Group Series can be accessed. The goal is thus to determine the function for each value of $g$. Two approaches to this are utilized in this paper, called the bottom-up and the top-down approach respectively.

In the bottom-up approach, the goal is to obtain the first member of the group, and then obtain the function to generate the sum with the remaining members of the group. It can be seen that each first member of the group can be obtained as $g \cdot (n - 1) + 1$, where $n = 1, 2, 3, 4, \ldots$. The group series can thus be displayed as $T_g(n) = (g \cdot (n - 1) + 1 + 0) + (g \cdot (n - 1) + 1 + 1) + (g \cdot (n - 1) + 1 + 2) + \cdots + (g \cdot (n - 1) + 1 + (g - 1))$. Since there will be $g$ instances of $g \cdot (n - 1) + 1$, the function will thus be $T_g(n) = g \cdot (g \cdot (n - 1) + 1) + (0 + 1 + 2 + 3 + \cdots + (g - 1))$. Since the term in the bracket to the right-hand side is a partial sum of the Triangular Number Series from 1 up to $g - 1$, it is equal to $\frac{(g-1)\cdot g}{2}$, so $T_g(n) = g^2 \cdot n - g^2 + g + \frac{(g-1)\cdot g}{2}$, or $T_g(n) = g^2\left(n - \frac{1}{2}\right) + \frac{g}{2}$.

In the top-down approach, the goal is to obtain the last member of the group, and from it obtain the remaining members of the group and the respective sum which will give the value of the total group. The last member of the group is obtained as $g \cdot n$, , whereas the total value of the group including each other member can be displayed as $T_g(n) = g \cdot n - (g - 1) + g \cdot n - (g - 2) + \cdots + g \cdot n - 1 + g \cdot n - 0 = g \cdot g \cdot n - ((g - 1) + (g - 2) + \cdots + 1 + 0) = g^2 \cdot n - \frac{(g-1)\cdot g}{2} = g^2\left(n - \frac{1}{2}\right) + \frac{g}{2}$. So, the Triangular Number Group Series Function is $T_g(n) = g^2\left(n - \frac{1}{2}\right) + \frac{g}{2}$, whatever the approach.

## 3. The Ramanujan Summation Function Results

Given the Triangular Number Group Series Function, implementing the Ramanujan Summation Function upon it yields the following result, which is given as Equation (2):

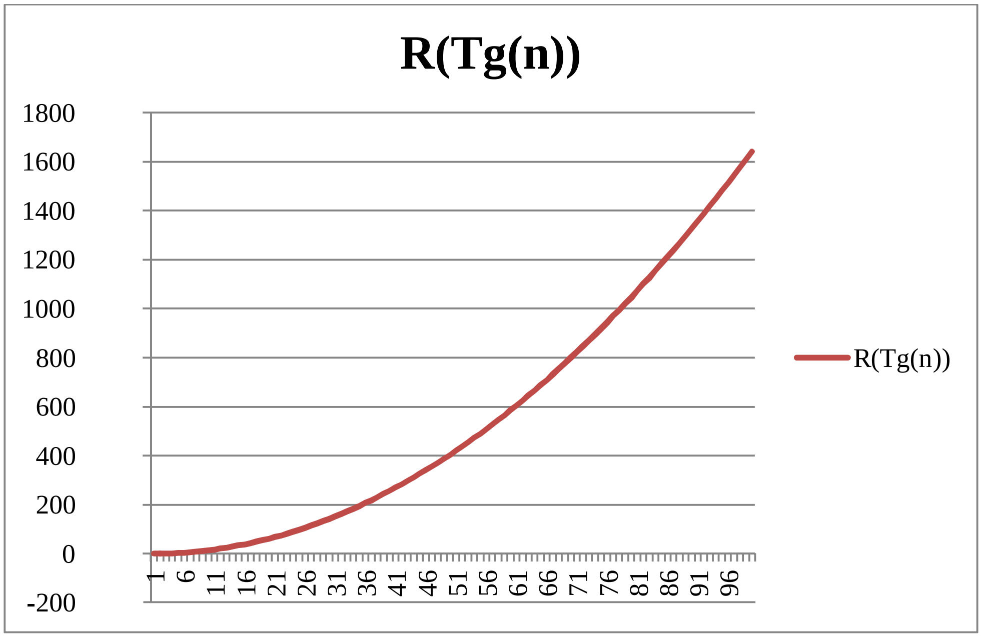
$$R\left(T_g(n)\right) = \frac{g^2}{6} - \frac{g}{4}. \tag{2}$$

Since, by definition, $T(n) = T_1(n)$, it yields that $R(T_1(n)) = -\frac{1}{12}$, so the obtained result is in accordance with the initial proposition that $R(1 + 2 + 3 + 4 + \cdots) = -\frac{1}{12}$. For sizes of the groups greater than 1, the result is consistent, i.e., solving the Ramanujan Summation Function given in Equation (1) and substituting $f(n)$ with $T_g(n)$ for any value of $g$ will give the result as in Equation (2). For example, $R\big(T_2(n)\big) = R\big((1+2)+(3+4)+(5+6)+\cdots\big) = R(3+7+11+\cdots) = \frac{1}{6}$ , $R\big(T_3(n)\big) = R\big((1+2+3)+(4+5+6)+(7+8+9)+\cdots\big) = R(6+15+24+\cdots) = \frac{3}{4}$ etc. Table 1 and Figure 1 give the results numerically and graphically, respectively, for the values of $g$ ranging from 1 to 100.

Table 1. Values of $R(T_g(n))$ for $g$ ranging from 1 to 100.

| $g$ | $R(T_g(n))$ | $g$ | $R(T_g(n))$ | $g$ | $R(T_g(n))$ | $g$ | $R(T_g(n))$ | $g$ | $R(T_g(n))$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | - 1/12 | 21 | 68 1/4 | 41 | 269 11/12 | 61 | 604 11/12 | 81 | 1073 1/4 |
| 2 | 1/6 | 22 | 75 1/6 | 42 | 283 1/2 | 62 | 625 1/6 | 82 | 1100 1/6 |
| 3 | 3/4 | 23 | 82 5/12 | 43 | 297 5/12 | 63 | 645 3/4 | 83 | 1127 5/12 |
| 4 | 1 2/3 | 24 | 90 | 44 | 311 2/3 | 64 | 666 2/3 | 84 | 1155 |
| 5 | 2 11/12 | 25 | 97 11/12 | 45 | 326 1/4 | 65 | 687 11/12 | 85 | 1182 11/12 |
| 6 | 4 1/2 | 26 | 106 1/6 | 46 | 341 1/6 | 66 | 709 1/2 | 86 | 1211 1/6 |
| 7 | 6 5/12 | 27 | 114 3/4 | 47 | 356 5/12 | 67 | 731 5/12 | 87 | 1239 3/4 |
| 8 | 8 2/3 | 28 | 123 2/3 | 48 | 372 | 68 | 753 2/3 | 88 | 1268 2/3 |
| 9 | 11 1/4 | 29 | 132 11/12 | 49 | 387 11/12 | 69 | 776 1/4 | 89 | 1297 11/12 |
| 10 | 14 1/6 | 30 | 142 1/2 | 50 | 404 1/6 | 70 | 799 1/6 | 90 | 1327 1/2 |
| 11 | 17 5/12 | 31 | 152 5/12 | 51 | 420 3/4 | 71 | 822 5/12 | 91 | 1357 5/12 |
| 12 | 21 | 32 | 162 2/3 | 52 | 437 2/3 | 72 | 846 | 92 | 1387 2/3 |
| 13 | 24 11/12 | 33 | 173 1/4 | 53 | 454 11/12 | 73 | 869 11/12 | 93 | 1418 1/4 |
| 14 | 29 1/6 | 34 | 184 1/6 | 54 | 472 1/2 | 74 | 894 1/6 | 94 | 1449 1/6 |
| 15 | 33 3/4 | 35 | 195 5/12 | 55 | 490 5/12 | 75 | 918 3/4 | 95 | 1480 5/12 |
| 16 | 38 2/3 | 36 | 207 | 56 | 508 2/3 | 76 | 943 2/3 | 96 | 1512 |
| 17 | 43 11/12 | 37 | 218 11/12 | 57 | 527 1/4 | 77 | 968 11/12 | 97 | 1543 11/12 |
| 18 | 49 1/2 | 38 | 231 1/6 | 58 | 546 1/6 | 78 | 994 1/2 | 98 | 1576 1/6 |
| 19 | 55 5/12 | 39 | 243 3/4 | 59 | 565 5/12 | 79 | 1020 5/12 | 99 | 1608 3/4 |
| 20 | 61 2/3 | 40 | 256 2/3 | 60 | 585 | 80 | 1046 2/3 | 100 | 1641 2/3 |

Figure 1. A graph of the values of $R(T_g(n))$ for $g$ ranging from 0 to 100




# 4. Discussion

## 4.1. Dependence of the Ramanujan Summation Function on g

Equation (2) and the results shown in Table 1 and Figure 1 show that, for divergent sums, it is relevant how the members of the series are arranged, since the results obtained from a single sum may be dependent upon how the member generating function of that sum is defined. Put differently, while the rearrangement of the terms of convergent sums always leads to the same value associated with that sum, different rearrangements of divergent sums lead to different values associated with them. This knowledge is confirmed, in the case of this paper, with the Ramanujan Summation Function results obtained from the Triangular Number Group Series Function, where an additional formalization of the Function results' dependence on parameter $g$ is established.

## 4.2. Extension of the Scope of g

The parameter $g$ in the Triangular Number Series Function $T_g(n)$ is the number of elements from the original Triangular Number Series, the (partial) sum of which forms the element of $T_g(n)$ with index $n$. As such, it makes sense that $g$ be a natural number. However, once Equation (2) is obtained, i.e., a generalized result involving $g$ is available, the scope of $g$ can be extended to real and even complex numbers. Figure 2 shows the results for $R(T_g(n))$ for values of $g$ from $-100$ to $100$ when $g$ is considered a real number, whereas Figure 3 shows the results for $R(T_g(n))$ for values of $g$ ranging from $-100 - i \cdot 100$ to $100 + i \cdot 100$, i.e., when $g$ is considered a complex number.

Figure 2. The function $R(T_g(n))$ for real values of $g$ from $-100$ to $100$ [14]

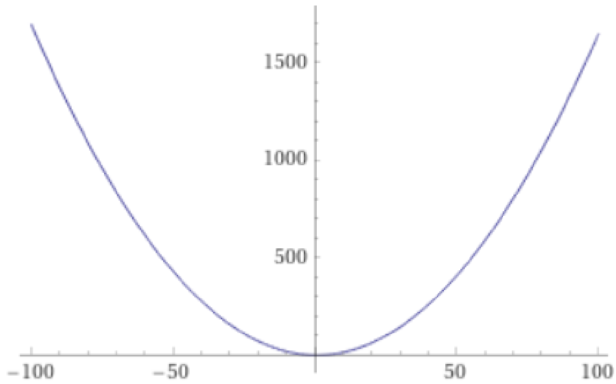
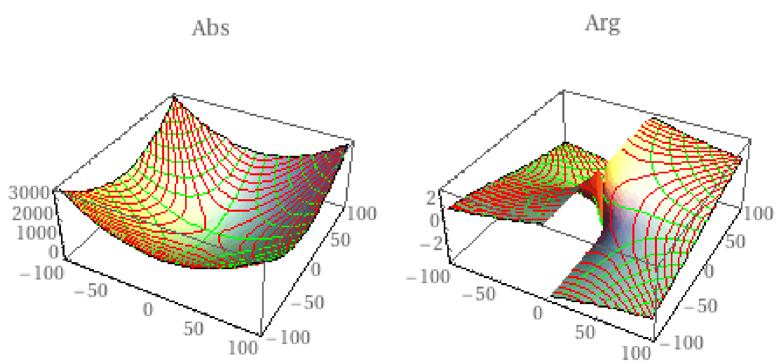



Figure 3. The function $R(T_g(n))$ for complex values of $g$ from $-100 - i \cdot 100$ to $100 + i \cdot 100$, shown with its real part (Abs) and imaginary part (Arg) [15]




# 5. Conclusion

In this paper, the Ramanujan Summation Function has been defined as a function that can be used upon a sum, with an appropriate member generating function. Such a generating function used in this paper is the Triangular Number Group Series Function, in which the values of its members depend upon the parameter $g$, which represents the number of elements from the Triangular Number Series. Depending on the value of $g$, different divergent sums are obtained from the Triangular Number Series, but it is shown that they follow a certain regularity, having $g$ as their parameter. Expanding the scope of $g$ to real and complex values is also possible, and the corresponding results of doing so are given as well.

## References

[1] Andrews, G.C., Berndt, B., 2005, *Ramanujan's Lost Notebook: Part I,* Springer

[2] Andrews, G.C., Berndt, B.C., 2009, *Ramanujan's Lost Notebook: Part II*, Springer

[3] Andrews, G.C., Berndt, B.C., 2012, *Ramanujan's Lost Notebook: Part III*, Springer

[4] Andrews, G.C., Berndt, B.C., 2013, *Ramanujan's Lost Notebook: Part IV*, Springer

[5] Andrews, G.C., Berndt, B.C., 2018, *Ramanujan's Lost Notebook: Part V*, Springer

[6] Candelpherger, B., 2017, *Ramanujan Summation of Divergent Series*, Springer

[7] Chagas, C.Q., Tenreiro Machado, J.A., Lopes, A.M., 2021, "Overview in Summabilities: Summation Methods for Divergent Series, Ramanujan Summation and Fractional Finite Sums", Mathematics, Vol. 9, No. 2963, DOI: 10.3390/math9222963

[8] Numberphile, 2014, "ASTOUNDING: 1 + 2 + 3 + 4 + 5 + ... = -1/12", https://www.youtube.com/watch?v=w-I6XTVZXww, last accessed 13 of July 2021.

[9] Numberphile, 2015, "Sum of Natural Numbers (second proof and extra footage)", https://www.youtube.com/watch?v=E-d9mgo8FGk, last accessed 13 July 2021.

[10] OEIS, 2021, "A000217 – Triangular numbers: a(n) = binomial(n+1,2) = n*(n+1)/2 = 0 + 1 + 2 + ... + n. (Formerly M2535 N1002)", Online Encyclopedia of Integer Sequences, https://oeis.org/A000217, last accessed 13 July 2021.

[11] PhysicsCentral, 2014, "Correction: Does 1+2+3+4+ . . . =-1/12? Absolutely Not! (I think)", http://physicsbuzz.physicscentral.com/2014/01/redux-does-1234-112-absolutely-not.html, last accessed 13 July 2021.

[12] Smail, L.L., 1919, "A General Method of Summation of Divergent Series", Annals of Mathematics, Second Serues, Vol. 20, No. 3, pp. 149-154.

[13] Schultz, C., 2014, "The Great Debate Over Whether 1+2+3+4..+ ∞ = -1/12", Smithsonian Magazine, https://www.smithsonianmag.com/smart-news/great-debate-over-whether-1234-112-180949559/, last accessed 13 July 2021.

[14] WolframAlpha, 2021a, "plot g^2/6 – g/4 from -100 to 100", https://www.wolframalpha.com/input/?i=plot+g%5E2%2F6+-+g%2F4+from+-100+to+100, last accessed on 13 July 2021.

[15] WolframAlpha, 2021b, "plot g^2/6 – g/4 from -100-100*i to 100+100*i", https://www.wolframalpha.com/input/?i=plot+g%5E2%2F6+-+g%2F4+from+-100-100*i+to+100%2B100*i, last accessed on July 13, 2021.

[16] zachx, 2015, "1+2+3+4+...=-1/12 weirdness", https://www.physicsforums.com/threads/1-2-3-4-1-12-weirdness.817600/, last accessed 13 July 2021.