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Norm equalities and inequalities for operator matrices

Research Article

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Abstract: The aim of this thesis is to study some Norm equalities and Inequalities for operator Matrices and some of their properties. Also, we present the aim of this thesis to present some famous norm inequalities for Hilbert space operators which have been studied in literature and some new results.

Keywords: Numerical radius • Numerical range • Spectral radius • Hilbert space • Cauchy-Schwarz inequality.

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1. Introduction

Over the last few decades, linear algebra has been increasingly popular. This subject appeals to people because of its beauty and ties to a variety of other pure and applied fields. In both the theoretical and practical evolution of the subject In many applications, it is necessary to determine the “length” of vectors. Norm is used for this purpose. On a vector space, functions are considered. We explain why one should do so in this informative essay. would like to investigate several types of norms in a real vector space.

A matrix norm is a measurement of the size of its elements. It is a method of determining the ”size” of a matrix that is unrelated to the number of rows or columns in the matrix.

A matrix norm is a real integer that represents the magnitude of the matrix. We will consider the vector spaces $M_n(R)$ and $M_n(C)$ of square $n \times n$ matrices for the sake of clarity.

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The majority of the conclusions are likewise valid for the $B(H)$ and $B(H)$ spaces of rectangular mn matrices. The rationale behind matrix norms is that they should perform "good" when it comes to matrix multiplication because $n \times n$ matrices can be multiplied.

Lemma 1.1.
[1] Let $M, N \in LB(H)$. Then

$$(i)\ \omega\left(\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}\right) = \max\{\omega(M), \omega(N)\},$$

$$(ii)\ \omega\left(\begin{bmatrix} M & N \\ N & M \end{bmatrix}\right) = \max\{\omega(M+N), \omega(M-N)\},$$

$$(iii)\ \omega\left(\begin{bmatrix} M & N \\ -N & M \end{bmatrix}\right) = \max\{\omega(M+iN), \omega(M-iN)\}.$$

Theorem 1.1.
[?] Let $\alpha_1, \ldots, \alpha_n \in K$ and $A_1, \ldots, A_n \in B(H)$. Then one has the inequalities:

$$\left|\left|\sum_{i=1}^{n} \alpha_i A_i\right|\right|^2 \leq \begin{cases} \max\limits_{i=1,\ldots,n} |\alpha_i|^2 \sum\limits_{i=1}^{n} ||A_i||^2 \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2p}\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} ||A_i||^{2q}\right)^{\frac{1}{q}} & \text{if } p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1; \\ \sum\limits_{i=1}^{n} |\alpha_i|^2 \max\limits_{i=1,\ldots,n} ||A_i||^2 \end{cases}$$
$$+ \begin{cases} \max\limits_{1 \leq i \neq j \leq n} \{|\alpha_i||\alpha_j|\} \sum\limits_{1 \leq i \neq j \leq n} ||A_i A_j^*|| \\ \left[\left(\sum\limits_{i=1}^{n} |\alpha_i|^r\right)^2 - \sum\limits_{i=1}^{n} |\alpha_i|^{2r}\right]^{\frac{1}{r}} \left(\sum\limits_{1 \leq i \neq j \leq n} ||A_i A_j^*||^s\right)^{\frac{1}{s}} & \text{if } r > 1,\ \dfrac{1}{r} + \dfrac{1}{s} = 1; \\ \left[\left(\sum\limits_{i=1}^{n} |\alpha_i|\right)^2 - \sum\limits_{i=1}^{n} |\alpha_i|^2\right] \max\limits_{1 \leq i \neq j \leq n} ||A_i A_j^*||, \end{cases} \tag{1}$$

where (1) should be seen as all the 9 possible configurations.

Corollary 1.1.
[?] With the assumptions in Theorem 1.1, one has the inequality:

$$\left|\left|\sum_{i=1}^{n} \alpha_i A_i\right|\right|^2 \leq \left(\sum_{i=1}^{n} |\alpha_i|^{2p}\right)^{\frac{1}{p}} \left[\left(\sum_{i=1}^{n} ||A_i||^{2q}\right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} ||A_i A_j^*||^q\right)^{\frac{1}{q}}\right], \tag{2}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1.1.
[3] For the case of two operators, we can state that

$$||B + C||^2 \leq ||B^* B + C^* C|| + \frac{1}{2}||B^* C + C^* B||^2 + \frac{1}{2} \tag{3}$$

for any B, $C \in B(H)$. If in this inequality we choose $B = A$, $C = A^*$, then we get

$$||A + A^*||^2 \leq ||A^*A + AA^*|| + \frac{1}{2}||A^2 + (A^*)^2||^2 + \frac{1}{2} \tag{4}$$

for any $A \in B(H)$.
Now, if $A = B + C$ with $B = (A + A^*)/2$, $C = (A - A^*)/2$, i.e., B and C are the Cartesian decomposition of A, then applying (3) for B and C as above will give the inequality

$$||A||^2 \leq \frac{1}{2}||A^*A + AA^*|| + \frac{1}{4}||A^*A - AA^*||^2 + \frac{1}{2} \tag{5}$$

for any $A \in B(H)$.

Theorem 1.2.
[3] For any two bounded linear operators C, D we have

$$w(CD) \leq 4w(C)w(D). \tag{6}$$

If C and D commute, i.e., $CD = DC$, then

$$w(CD) \leq 2w(C)w(D). \tag{7}$$

To get closer to the elusive inequality

$$w(CD) \leq w(C)w(D). \tag{8}$$

Theorem 1.3.
[6] Let $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in M_2(M_n)$ is positive semidefinite. Then

$$|trAC - trB^*B| \leq trAtrC - |trB|^2. \tag{9}$$

2. Main Results

In this section, we present some new results inequalities for operator Matrices.

Theorem 2.1.
If M, N are normal matrices, then

1. $\left|\left|\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}\right|\right| = \max\{||M||, ||N||\}$.
2. $\left|\left|\begin{bmatrix} M & N \\ N & M \end{bmatrix}\right|\right| = \max\{||M + N||, ||M - N||\}$.

Proof. 1. Since, M and N are Hermitian matrices; M and N are normal matrices. So

$$W(M) = ||M|| \quad \text{and} \quad W(N) = ||N||.$$

also, from Lemma 1.1; we have

$$W\left(\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}\right) = \max\{W(M), W(N)\}$$

and the matrix $\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ is Hermitian matrix

$$W\left(\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}\right) = \left\|\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}\right\|$$

So, we get the result.

2. Suppose M, N are Hermitian, then $M+N$ and $M-N$ are Hermitian by using the similar way of (1) we obtain (2).

□

Now; we present special case for theorem (1.1) as follows.

Theorem 2.2.
Let $A_1, \dots, A_n \in B(H)$. Then one has the inequalities

$$\left\|\sum_{i=1}^{n} A_i\right\|^2 \le \begin{cases} \max\limits_{i=1,\dots,n} \sum\limits_{i=1}^{n} ||A_i||^2 \\ \left(\sum\limits_{i=1}^{n} ||A_i||^4\right)^{\frac{1}{2}} \\ \sum\limits_{i=1}^{n} \max\limits_{i=1,\dots,n} ||A_i||^2 \end{cases}$$

$$+ \begin{cases} \max\limits_{1<i\neq j<n} \sum\limits_{1<i\neq j\leq n} ||A_i A_j^*|| \\ [n^2-n] \left(\sum\limits_{1\leq i\neq j\leq n} ||A_i A_j^*||^2\right)^{\frac{1}{2}} \\ [n^2-n] \max\limits_{1\leq A_i A_j^* \leq n} ||A_i A_j^*|| \end{cases}$$

Proof. We know that by Theorem (1.1)

$$\left|\left|\sum_{i=1}^{n}\alpha_i A_i\right|\right|^2 \le \begin{cases} \max\limits_{i=1,\ldots,n}|\alpha_i|^2\sum\limits_{i=1}^{n}||A_i||^2 \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{2p}\right)^{\frac{1}{p}}\left(\sum\limits_{i=1}^{n}||A_i||^{2q}\right)^{\frac{1}{q}} & \text{if } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \sum\limits_{i=1}^{n}|\alpha_i|^2\max\limits_{i=1,\ldots,n}||A_i||^2 \end{cases}$$

$$+\begin{cases} \max\limits_{1\le i\ne j\le n}\{|\alpha_i||\alpha_j|\}\sum\limits_{1\le i\ne j\le n}||A_iA_j^*|| \\ \left[\left(\sum\limits_{i=1}^{n}|\alpha_i|^r\right)^2-\sum\limits_{i=1}^{n}|\alpha_i|^{2r}\right]^{\frac{1}{r}}\left(\sum\limits_{1\le i\ne j\le n}||A_iA_j^*||^s\right)^{\frac{1}{s}} \\ \quad \text{if } r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \left[\left(\sum\limits_{i=1}^{n}|\alpha_i|\right)^2-\sum\limits_{i=1}^{n}|\alpha_i|^2\right]\max\limits_{1\le i\ne j\le n}||A_iA_j^*||, \end{cases}$$

By letting $\alpha_i = 1\ \forall\ i = 1,\ldots,n$ and $p = q = r = s = 2$ we get

$$\left|\left|\sum_{i=1}^{n}A_i\right|\right|^2 \le \begin{cases} \max\limits_{i=1,\ldots,n}\sum\limits_{i=1}^{n}||A_i||^2 \\ \left(\sum\limits_{i=1}^{n}||A_i||^4\right)^{\frac{1}{2}} \\ \sum\limits_{i=1}^{n}\max\limits_{i=1,\ldots,n}||A_i||^2 \end{cases}$$

$$+\begin{cases} \max\limits_{1<i\ne j<n}\sum\limits_{1<i\ne j\le n}||A_iA_j^*|| \\ [n^2-n]\left(\sum\limits_{1\le i\ne j\le n}||A_iA_j^*||^2\right)^{\frac{1}{2}} \\ [n^2-n]\max\limits_{1\le A_iA_j^*\le n}||A_iA_j^*|| \end{cases}$$

□

In the following corollary we present the special case for Theorem (2.2).

Corollary 2.1.
By taking $p = q = 2$, $\alpha i = 1\ \forall\ i = 1,\ldots,n$ in corollary (1.1) we get the inequality

$$\left|\left|\sum A_i\right|\right|^2 \le \sqrt[p]{n}\left[\left(\sum_{i=1}^{n}||A_i||^4\right)^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}\left(\sum||A_iA_i^*||^2\right)^{\frac{1}{2}}\right]$$

Theorem 2.3.
Let $A_1, \ldots, A_n \in B(H)$. Then one has the inequalities

$$\left\|\sum_{i=1}^{n} 2A_i\right\|^2 \leq 4\sum_{j=1}^{n} ||A_i A_j^*||$$
$$\leq \begin{cases} 4 \max\limits_{1<i<n} \left[\sum\limits_{j=1}^{n} ||A_i A_j^*||\right] \\ 8\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1} ||A_i A_j^*||\right)^2\right]^{\frac{1}{2}} \end{cases}$$

Proof. We know that by theorem (**??**)

$$\left\|\sum_{i=1}^{n} \alpha_i A_i\right\|^2 \leq \sum_{i=1}^{n} |\alpha_i|^2 \sum_{j=1}^{n} ||A_i A_j^*||$$
$$\leq \begin{cases} \sum\limits_{i=1}^{n} |\alpha_i|^2 \max\limits_{1\leq i\leq n} \left[\sum\limits_{j=1}^{n} ||A_i A_j^*||\right]; \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2p}\right)^{\frac{1}{p}} \left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} ||A_i A_j^*||\right)^q\right]^{\frac{1}{q}} & \text{where } p > 1,\ \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \max\limits_{1\leq i\leq n} |\alpha_i|^2 \sum\limits_{i,j=1}^{n} ||A_i A_j^*||. \end{cases}$$

By letting $\alpha i = 2$ and $p = q = 2\ \forall\ i = 1, \ldots, n$

$$\left\|\sum_{i=1}^{n} 2A_i\right\|^2 \leq 4\sum_{j=1}^{n} ||A_i A_j^*||$$
$$\leq \begin{cases} 4 \max\limits_{1<i<n} \left[\sum\limits_{j=1}^{n} ||A_i A_j^*||\right] \\ 8\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1} ||A_i A_j^*||\right)^2\right]^{\frac{1}{2}} \end{cases}$$

□

Theorem 2.4.
If $A \in M_n(C)$, then

$$||A|| \leq ||I + A^*A|| + \frac{1}{2}||A + A^*|| + \frac{1}{2}$$

Proof. from the inequality (3) we have

$$||B + C||^2 \leq ||B^*B + C^*C|| + \frac{1}{2}||B^*C + C^*B||^2 + \frac{1}{2}$$

By letting $B = I$, $C = A$ in (**??**) we get

$$||I + A|| \leq ||I + A^*A|| + \frac{1}{2}||A + A^*|| + \frac{1}{2}$$

We know also $||A|| \leq ||I + A||$

So,

$$||A|| \leq ||I + A^*A|| + \frac{1}{2}||A + A^*|| + \frac{1}{2}$$

□

Corollary 2.2.
If we put $A = I$ in (5) and taking $||A|| = ||A||_1$, we have after some calculations $1 \leq \frac{1}{2}||2I|| + \frac{1}{2}$ and hence $1 \leq \frac{3}{2}$.

Theorem 2.5.
If A, B are double commute i.e $AB = BA$ and $AB^* = B^*A$, then

$$w(AB) \leq w(B)S_i(A)$$

Proof. If $||A|| = ||A||_2 = S_i(A)$ and the result follows by putting $||A|| = S_i(A)$ in (8). □

Corollary 2.3. *1. For any two bound linear operator A, B we have*

$$w(AB) \leq 4||A||\ ||B||$$

2. If $A, B \in M_n(C)$ and $AB = BA$, then

$$w(AB) \leq 2||A||\ ||B||$$

To illustration this Remark we know that $w(A) \leq ||A||\ \forall\ A \in M_n(C)$ and using inequality (6) and (7).

Lemma 2.1.
If $A, B \in B(H)$, then

1. $r(A) = ||A^{\frac{1}{2}}||^2 \leq ||A||$
2. $||A^{\frac{1}{2}}B^{\frac{1}{2}}||^2 \leq w(AB) \leq ||AB||$

Proof. 1. we know that

$$r(AB) = ||A^{\frac{1}{2}}B^{\frac{1}{2}}||^2$$

put $B = I$;

$$r(A) = ||A^{\frac{1}{2}}||^2$$

also;

$$||A^n||^m \leq ||A^{nm}|| \quad \forall\ n, m \in R$$

so;

$$r(A) = ||A^{\frac{1}{2}}||^2 \leq ||A||$$

2.

$$r(AB) = ||A^{\frac{1}{2}}B^{\frac{1}{2}}||^2 \leq w(AB)$$

because $r(AB) \leq w(AB) \ \forall \ A, B \in B(H)$ also; we know that $w(AB) \leq ||AB||$ so,

$$||A^{\frac{1}{2}}B^{\frac{1}{2}}||^2 \leq w(AB) \leq ||AB||$$

□

Theorem 2.6.
Let $\begin{bmatrix} A & I \\ I & C \end{bmatrix} \in M_n(C)$ *is positive semidefinite. Then*

$$|tr \ AC - n| \leq tr \ A \ tr \ C - n^2.$$

Proof. By Theorem 1.3 we have;

$$|tr \ AC - trB^*B| \leq tr \ A \ tr \ C - |tr \ B|^2$$

and by letting $B = I = B^*$, we have

$$|trAC - tr \ I| \leq tr \ A \ tr \ C - |tr \ I|^2$$

and since $tr \ I = n$ we get the result.

□

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