

Stability analysis and synchronization of incommensurate fractional-order neural networks

Research Article

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Abstract: This paper develops a theoretical framework for analyzing the stability of nonlinear incommensurate fractional-order neural networks. A necessary theorem for asymptotical stability is established using the characteristic equation for a nonlinear fractional-order system, and how to employ this theorem in stabilization is also presented. With the suitable control, the difficulties of stabilization and synchronization of fractional-order chaotic incommensurate fractional-order neural networks may be readily overcome. Two numerical examples have been shown to demonstrate how the established theory may be used to investigate stability and construct stabilization controllers.

Keywords: Fractional calculus • Incommensurate fractional-order neural networks • Asymptotic stability • Synchronization

1. Introduction

The study of differential and integral operators of real and complex order referred to as fractional calculus was developed by many famous mathematicians such as Liouville, Riemann, Abel, and Caputo. Even though the branch of fractional calculus began practically concurrently with its integer-order equivalent, the mathematics and, in particular, its applications are somewhat less developed, and several causes have led to this conclusion. The fact that these operators are multiscale by nature is noteworthy, as a consequence, memory effects are enabled by time-fractional operators. In the last few decades, fractional calculus has developed fast and has many applications in a wide range of areas [13][8][2][18][1]. It has been revealed that fractional differential equations are strongly connected to systems having memory and hereditary features [20][21][11]. Furthermore, it offers an

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efficient description approach for mathematical models, as an example secure communication [2], control fields [23], digital signal processor [17] and so on.

In recent years, neural networks have received a great deal of attention due to their effective applications in a wide range of fields, including optimization, function approximation, associative memory, signal processing, automated control, and etc [7][24][22]. These applications rely significantly on the stability qualities of neural networks. As a result, the stability analysis of neural networks has received much interest in the research. To its memory and heredity features, for that fractional-order neural networks are projected to be particularly useful in many applications [6]. It is also shown that fractional-order neural networks may be successfully employed in a variety of study domains [14][4]. Strong requirements have been offered for commensurate fractional order neural networks (including derivation orders that are all multiples of the same order), in [27] Global asymptotic stability of fractional-order competitive neural networks with multiple time-varying-delay links was investigated, Mittag-Leffler stability and adaptive impulsive synchronization of fractional order neural networks in quaternion field was adressed in [3], and in [9], New criteria on finite-time stability of fractional-order hopfield neural networks with time delays was introduced, [15] treated α -stability and α -synchronization for fractional-order neural networks.

On the other hand, the synchronization problem initially introduced in [19], is a basic phenomenon that has recently received a lot of attention, since it has both theoretical and practical implications. One of the key issues is to accomplish and explain synchronization of chaotic systems that are strictly distinct. As it has been demonstrated that certain neural networks can manifest chaotic behavior, different synchronization results and approaches regarding constant fractoinal order neural networks have been published in the previous decade [12][25][10][28][29][30]. Because of the complicated dynamics of incommensurate systems, these conclusions and approaches could not be simply extended and applied to the incommensurate case.

All of the articles mentioned above studied the commensurate fractional-order neural networks, which indicates that the conclusions achieved may be invalid for incommensurate fractional-order neural networks (this includes all derivation orders that are multiples of different order). Because commensurate fractional-order neural networks are a subtype of incommensurate fractional-order neural networks, it is crucial to explore the stability and synchronization of incommensurate fractional-order neural networks. However, relatively few related results concerning these systems have been reported so far. Motivated by the earlier discussion, in this work we address the stability and synchronization of incommensurate fractional-order neural networks.

This paper's reminder is arranged as follows. Section 2 contains some preliminaries, such as definitions, lemma and theorems to Caputo fractional-order calculation. In Section 3.1, numerous necessary conditions for determining the stability criteria of incommensurate fractional-order neural networks are developed. Some essential conditions for synchronization of incommensurate fractional-order neural networks are derived in Section 3.2. Section 4 provides two numerical simalations to demonstrate the efficacy of the stated results. Fanyilly, Section 5 provides the conclusion to this study.

2. Preliminaries

Definition 2.1 ([8]).

The fractional-order integral of order α for an integrable function $f : [t_0, +\infty) \rightarrow \mathbb{R}$ is defined as

$${}_{t_0}I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t - \tau)^{\alpha-1} f(\tau) d\tau,$$

where $\alpha > 0$, and $\Gamma(\cdot)$ is the Gamma function which is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad (Re(z) > 0)$$

where $Re(z)$ is the real part of z .

Definition 2.2 ([8]).

The Caputo fractional-order derivative of order α for a function f is defined as

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau,$$

where $t \geq t_0$ and n is a positive integer such that $n - 1 < \alpha < n$.
 particularly, when $0 < \alpha < 1$

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_{t_0}^t (t - \tau)^{-\alpha} f'(\tau) d\tau,$$

Proposition 2.1 ([8]). • ${}^C D_t^\alpha c = 0$ holds, where c is any constant.

• The linearity of the Caputo fractional order derivative:

$${}^C D_t^\alpha (v_1 f(t) + v_2 g(t)) = v_1 {}^C D_t^\alpha f(t) + v_2 {}^C D_t^\alpha g(t)$$

Lemma 2.1 ([8]).

Let $n = [\alpha] + 1$ for α or $n = \alpha$ for $\alpha \in \mathbb{N}_+$, if $x(t) \in C^n[a, b]$ then,

$${}^{RL} I_t^\alpha {}^C D_t^\alpha x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!} (t - t_0)^k, \quad n - 1 < \alpha \leq n.$$

In particular if $0 < \alpha \leq 1$ and $x(t) \in C^1[a, b]$, then

$${}^{RL} I_t^\alpha {}^C D_t^\alpha x(t) = x(t) - x(t_0)$$

Theorem 2.1 ([26]).

Consider the following n -dimensional linear fractional order system:

$${}_0^C D_t^{\bar{\alpha}} x(t) = Ax(t) \tag{1}$$

where $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$, $A = (a_{ij})_{n \times n}$. Assume M be the lowest common multiple of the denominators u_i 's of α_i 's, where $\alpha_i = \frac{v_i}{u_i}$, $(u_i, v_i) = 1$, $u_i, v_i \in \mathbb{Z}^+$, for $i = 1, 2, \dots, n$.

Then the zero solution of system (1) is globally asymptotically stable in the Lyapunov sense if all roots λ_i 's of

$$\det \{ \text{diag} ([\lambda^{M\alpha_1} \quad \lambda^{M\alpha_2} \quad \dots \quad \lambda^{M\alpha_n}]) - A \} = 0$$

satisfy $|\arg(\lambda)| > \frac{\pi}{2M}$

Theorem 2.2 ([5]).

Consider the following n -dimensional linear fractional order system:

$${}^C_0 D_t^{\bar{\alpha}} x(t) = f(x(t)) \quad (2)$$

where $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$, $f(x(t)) = (f_1(x(t)), \dots, f_n(x(t)))$ is a nonlinear continuous function. Assume M be the lowest common multiple of the denominators u_i 's of α_i 's, where $\alpha_i = \frac{v_i}{u_i}$, $(u_i, v_i) = 1$, $u_i, v_i \in \mathbb{Z}^+$, for $i = 1, 2, \dots, n$.

Then the zero solution of system (2) is globally asymptotically stable in the Lyapunov sense if all roots λ_i 's of

$$\det \{ \text{diag} ([\lambda^{M\alpha_1} \quad \lambda^{M\alpha_2} \quad \dots \quad \lambda^{M\alpha_n}]) - J \} = 0$$

satisfy $|\arg(\lambda)| > \frac{\pi}{2M}$

where $J = \frac{\partial f}{\partial x}|_{x=x^*}$ is the jacobian matrix of f

3. Mains results

The incommensurate fractional-order neural networks can be described as follow

$$\left\{ \begin{array}{l} {}^C_{t_0} D_t^{\alpha_1} x_1(t) = -c_1 x_1(t) + \sum_{j=1}^n b_{1j} g_j(x_j(t)) + I_1 \\ {}^C_{t_0} D_t^{\alpha_2} x_2(t) = -c_2 x_2(t) + \sum_{j=1}^n b_{2j} g_j(x_j(t)) + I_2 \\ \dots \\ {}^C_{t_0} D_t^{\alpha_n} x_n(t) = -c_n x_n(t) + \sum_{j=1}^n b_{nj} g_j(x_j(t)) + I_n \\ x(t_0) = x_0 \end{array} \right. \quad (3)$$

or it can be written as

$$\left\{ \begin{array}{l} {}^C_{t_0} D_t^{\bar{\alpha}} x(t) = -Cx(t) + Bg(x(t)) + I \\ x(t_0) = x_0 \end{array} \right.$$

where ${}^C_{t_0} D_t^{\alpha_i}$, denotes the Caputo fractional derivative with order $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$, $(0 < \alpha_i < 1)$ and n represents the number of units in a neural network. $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^t \in \mathbb{R}$ is the state vector at time t . $(c_i) \in \mathbb{R}^{n \times n}$, $i \in \{1, 2, \dots, n\}$ with $c_i > 0$ is a diagonal matrix referred to the self-feedback connection weight, $(b_{ij}) \in \mathbb{R}^{n \times n}$ corresponds to the connection of i th neuron to the j th neuron at time t . $g : C([t_0, T], \mathbb{R}^n) \rightarrow \mathbb{R}^n$, $g(t, x(t)) = (g_1(t, x(t)), g_2(t, x(t)), \dots, g_n(t, x(t)))^T$ is the neuron continuous activation function and $I = (I_1, I_2, \dots, I_n)^T$ is the external inputs vector.

3.1. Stability of incommensurate fractional-order neural networks

Theorem 3.1.

Suppose that M is the lowest common multiple of the denominators u_i 's of α_i 's, where $\alpha_i = \frac{v_i}{u_i}$, $(u_i, v_i) = 1$, $u_i, v_i \in \mathbb{Z}^+$, for $i = 1, 2, \dots, n$. the equilibrium point $x = x^*$ of system (3) is asymptotically stable if

$$|\arg(\lambda)| > \frac{\pi}{2M},$$

for all roots λ 's of the following equation

$$\det \{ \text{diag} ([\lambda^{M\alpha_1} \quad \lambda^{M\alpha_2} \quad \dots \quad \lambda^{M\alpha_n}]) - (-C + BJ) \} = 0$$

where

$$J = \frac{\partial g}{\partial x} \Big|_{x=x^*}, \quad g = [g_1, g_2, \dots, g_n]^T$$

Proof. Solving the following equations yields the equilibrium points of system (3):

$$-c_i x_i(t) + \sum_{j=1}^n b_{ij} g_j(t, x_j(t)) + I_i = 0, \quad i = 1, 2, \dots, n \quad (4)$$

Suppose that $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ is an equilibrium point of system (3),

i.e.

$$-c_i x_i^* + \sum_{j=1}^n b_{ij} g_j(t, x_j^*) + I_i = 0; \quad i = 1, 2, \dots, n \quad (5)$$

To analyze this point's asymptotic stability, we establish

$$\kappa_i = x_i - x_i^*, \quad i = 1, 2, \dots, n$$

then we have

$${}_t_0^C D_t^{\alpha_i} (\kappa_i(t) + x_i^*) = -c_i (\kappa_i(t) + x_i^*) + \sum_{j=1}^n b_{ij} g_j(t, (\kappa_j(t) + x_j^*)) + I_i = 0, \quad i = 1, 2, \dots, n \quad (6)$$

which implies according to Lemma(2.1)

$${}_t_0^C D_t^{\alpha_i} \kappa_i(t) = -c_i (\kappa_i(t) + x_i^*(t)) + \sum_{j=1}^n b_{ij} g_j(t, (\kappa_j(t) + x_j^*(t))) + I_i = 0, \quad i = 1, 2, \dots, n$$

If function f_i has second continuous partial derivatives in a ball \mathbb{R}^n centered at point x^* , then we obtain

$$g_j(\kappa_1(t) + x_1^*, \kappa_2(t) + x_2^*, \dots, \kappa_n(t) + x_n^*) = g_j(x_1^*, x_2^*, \dots, x_n^*) + \left[\frac{df_1}{dx_1} \quad \frac{df_2}{dx_2} \quad \dots \quad \frac{df_n}{dx_n} \right] \kappa + g_j(\kappa)$$

therefor (6) becomes

$${}_t_0^C D_t^{\alpha_i} \kappa_i(t) = -c_i (\kappa_i(t) + x_i^*(t)) + \sum_{j=1}^n b_{ij} \left\{ g_j(x_j^*(t)) + \frac{df_j}{dx_j} \kappa_j(t) \right\} + I_i$$

which gives using (5)

$${}_t_0^C D_t^{\alpha_i} \kappa_i(t) = -c_i \kappa_i(t) + \sum_{j=1}^n b_{ij} \frac{df_j}{dx_j} \kappa_j(t)$$

and then

$$\begin{bmatrix} {}^C D_{t_0}^{\alpha_1} \kappa_1(t) \\ {}^C D_{t_0}^{\alpha_2} \kappa_2(t) \\ \dots \\ {}^C D_{t_0}^{\alpha_n} \kappa_n(t) \end{bmatrix} = (-C + BJ)\kappa$$

where

$$J = \frac{dg}{dx}, \quad g = [g_1, g_2, \dots, g_n]^T$$

Suppose that M is the lowest common multiple of the denominators u_i 's of α_i 's, where $\alpha_i = \frac{v_i}{u_i}$, $(u_i, v_i) = 1$, $u_i, v_i \in \mathbb{Z}^+$, for $i = 1, 2, \dots, n$. According to Theorem 2.2, the equilibrium point $x = x^*$ of system (3) is asymptotically stable if

$$|\arg(\lambda)| > \frac{\pi}{2M},$$

for all roots λ 's of the following equation

$$\det \left\{ \text{diag} \left(\left[\lambda^{M\alpha_1} \quad \lambda^{M\alpha_2} \quad \dots \quad \lambda^{M\alpha_n} \right] \right) - (-C + BJ) \right\} = 0$$

Which completes the proof. \square

3.2. Synchronization of incommensurate fractional order neural networks

In this section, the synchronization of incommensurate fractional order neural networks is investigated.

As known a required condition for a fractional order system ${}^C D_t^\alpha x(t) = f(x(t))$ to display the chaotic attractor is mathematically equal to:

$$\frac{\pi}{2M} - \min_i \{|\arg(\lambda_i)|\} \geq 0,$$

where λ_i 's are roots of equations:

$$\det \left\{ \text{diag} \left(\left[\lambda^{M\alpha_1} \quad \lambda^{M\alpha_2} \quad \dots \quad \lambda^{M\alpha_n} \right] \right) - \frac{df}{dx} \right\} = 0$$

Thus, under the following essential condition, incommensurate fractional-order neural networks (3) exhibit chaotic behavior if

$$\frac{\pi}{2M} - \min_i \{|\arg(\lambda_i)|\} \geq 0,$$

where λ_i 's are roots of equations:

$$\det \left\{ \text{diag} \left(\left[\lambda^{M\alpha_1} \quad \lambda^{M\alpha_2} \quad \dots \quad \lambda^{M\alpha_n} \right] \right) - (-C + BJ) \right\} = 0$$

Now, we consider a master system represented in (3) and a slave system given by:

$$\begin{cases} {}_0^C D_t^{\alpha_1} y_1(t) = -c_1 y_1(t) + \sum_{j=1}^n b_{1j} g_j(y_j(t)) + I_1 + U_1(t) \\ {}_0^C D_t^{\alpha_2} y_2(t) = -c_2 y_2(t) + \sum_{j=1}^n b_{2j} g_j(y_j(t)) + I_2 + U_2(t) \\ \dots \\ {}_0^C D_t^{\alpha_n} y_n(t) = -c_n y_n(t) + \sum_{j=1}^n b_{nj} g_j(y_j(t)) + I_n + U_n(t) \end{cases} \quad (7)$$

Definition 3.1 ([25]).

If any solutions $x(t)$ of (3) and $z(t)$ of (7) satisfy the condition

$$\lim_{t \rightarrow +\infty} \|x(t) - y(t)\| = 0$$

then systems (3) and (7) are said to be global asymptotic synchronization.

Theorem 3.2.

The master-slave pair (3)–(7) is globally complete synchronized by means of the control law:

$$U_i(t) = -[(-c_i + a_i)e_i(t) - c_i e_i(t) + \sum_{j=1}^n b_{ij}(g_j(y_j(t)) - g_j(x_j(t)))], \quad i = 1, \dots, n$$

subject to the selection of the control matrix A as follows

$$A = \text{diag}([a_1 \ \dots \ a_n]), \quad \text{such that} \quad -a_i + c_i > 0, \quad i = 1, \dots, n$$

Proof. Let's consider the following error

$$e_i(t) = y_i(t) - x_i(t), \quad i = 1, \dots, n$$

The Caputo fractional differential of the error system is given by

$${}_0^C D_t^{\alpha_i} e_i(t) = {}_0^C D_t^{\alpha_i} y_i(t) - {}_0^C D_t^{\alpha_i} x_i(t) = -c_i e_i(t) + \sum_{j=1}^n b_{ij}(g_j(y_j(t)) - g_j(x_j(t))) + U_i \quad (8)$$

Synchronization between master system (3) and slave system (7) is equivalent to the asymptotic stability of error system (8) with the suitable control law $U(t)$.

With the choose of

$$U_i = -[(-c_i + a_i)e_i(t) - c_i e_i(t) + \sum_{j=1}^n b_{ij}(g_j(y_j(t)) - g_j(x_j(t)))]$$

we obtain

$${}_0^C D_t^{\alpha_i} e_i(t) = (c_i - a_i)e_i(t)$$

Since ${}_0^C D_t^{\alpha_i} e_i(t) = (c_i - a_i)e_i(t)$ is a linear system according to Theorem (2.1) this system is stable iff

$$\det \left\{ \text{diag} \left(\left[\lambda^{M\alpha_1} \quad \lambda^{M\alpha_2} \quad \dots \quad \lambda^{M\alpha_n} \right] \right) - (C - A) \right\} = 0$$

therefor $-c_i + a_i > 0, \quad i = 1, \dots, n$

which accomplishes the proof. \square

4. Numerical examples

All simulations in this section were run using Adams–Bashforth–Moulton algorithm described in [16] to solve a Caputo definition-based fractional differential equation.

Example 1 We consider the incommensurate fractional-order neural networks

$$\begin{cases} {}^C D_t^{\alpha_1} x_1(t) = -c_1 x_1(t) + b_{11} \tanh(x_1(t)) + b_{12} \tanh(x_2(t)) + I_1 \\ {}^C D_t^{\alpha_2} x_2(t) = -c_2 x_2(t) + b_{21} \tanh\left(\frac{x_1(t)}{4}\right) + b_{22} \tanh\left(\frac{x_2(t)}{4}\right) + I_2 \end{cases} \quad (9)$$

where

$$B = \begin{bmatrix} -0.5 & 0.3 \\ -0.7 & 0.1 \end{bmatrix}; \quad C = \text{diag} \left(\begin{bmatrix} 0.1 & 0.2 \end{bmatrix} \right); \quad I = (0 \ 0)^T$$

and $(\alpha_1, \alpha_2) = (0.4, 0.3)$

To obtain the equilibrium points, let us consider the equations as

$$\begin{cases} -0.1x_1(t) - 0.5 \tanh(x_1(t)) + 0.3 \tanh(x_2(t)) = 0 \\ -0.2x_2(t) - 0.7 \tanh\left(\frac{x_1(t)}{4}\right) + 0.1 \tanh\left(\frac{x_2(t)}{4}\right) = 0 \end{cases} \quad (10)$$

After solving equations (10), we get the equilibrium point as $E(0, 0)$.

The Jacobian matrix for the function $g(x(t)) = [\tanh(x(t)), \tanh(\frac{x(t)}{4})]^T$ is given as

$$J = \begin{bmatrix} \frac{1}{\cosh^2(x_1(t))} & \frac{1}{\cosh^2(x_1(t))} \\ \frac{1}{4 \cosh^2(x_1(t))} & \frac{1}{4 \cosh^2(x_1(t))} \end{bmatrix}$$

at the equilibrium point $E(0, 0)$ the characteristic polynomial of the matrix $-C + BJ$ is given by

$$\Delta(\lambda) = \lambda^7 + 0.65\lambda^4 + 0.525\lambda^3 + 0.5325 \quad (11)$$

The roots λ_i 's and their appropriate arguments of polynomial of (11) calculated below

$$\begin{aligned} \lambda_{1,2} &= 0.5964 + 0.8154i, & |\arg(\lambda_1)| &= 0.9393 \\ \lambda_3 &= -0.8112 + 0.0000i, & |\arg(\lambda_3)| &= 3.1416 \\ \lambda_{4,5} &= -0.5589 + 0.4965i, & |\arg(\lambda_4)| &= 2.4153 \\ \lambda_{6,7} &= 0.3681 + 0.4703i, & |\arg(\lambda_6)| &= 0.9067 \end{aligned}$$

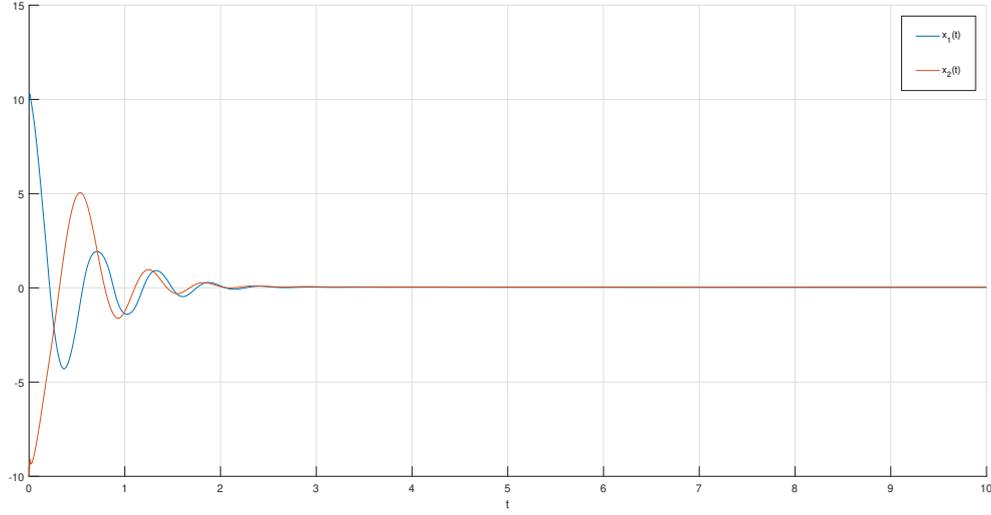


Figure 1. Numerical solution of the incommensurate neorder neural networks (9)

Since

$$|\arg(\lambda_i)| > \frac{\pi}{2M} = \frac{\pi}{20},$$

The incommensurate neural networks (9) is asymptotically stable.

For time step size 0.01 and initial value $x_0 = [10, -10]^T$, numerical simulation results in Figure 1 illustrate the asymptotically stability of system (9) for the given fractional-orders.

Exemple 2 Let's consider the three dimensionelle incommensurate fractional-order neural networks

$$\begin{cases} {}_0^C D_t^{\alpha_1} x_1(t) = -c_1 x_1(t) + b_{11} g(x_1(t)) + b_{12} g(x_2(t)) + b_{13} g(x_3(t)) + I_1 \\ {}_0^C D_t^{\alpha_2} x_2(t) = -c_2 x_2(t) + b_{21} g(x_1(t)) + b_{22} g(x_2(t)) + b_{23} g(x_3(t)) + I_2 \\ {}_0^C D_t^{\alpha_3} x_3(t) = -c_3 x_3(t) + b_{31} g(x_1(t)) + b_{32} g(x_2(t)) + b_{33} g(x_3(t)) + I_3 \end{cases} \quad (12)$$

where

$$B = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -4 & 1 \\ 5 & 1 & 2 \end{bmatrix}; \quad C = \text{diag}(0.8 \ 0.8 \ 0.8); \quad I = \begin{bmatrix} 0.01 & 0.01 & 0.01 \end{bmatrix}$$

$g(x(t)) = \sin(x(t))$ the initial condition $x_0 = (0.1, 0.1, 0.1)^T$ and $(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$ The equilibrium point is $x^*(0.0010, 0.0027, 0.0019)^T$ the characteristic polynomial of $-C + BJ$ matrix is given by

$$\Delta(\lambda) = \lambda^4 - 10.4\lambda^3 - 6.16\lambda^2 - 5.12\lambda - 5.888 \quad (13)$$

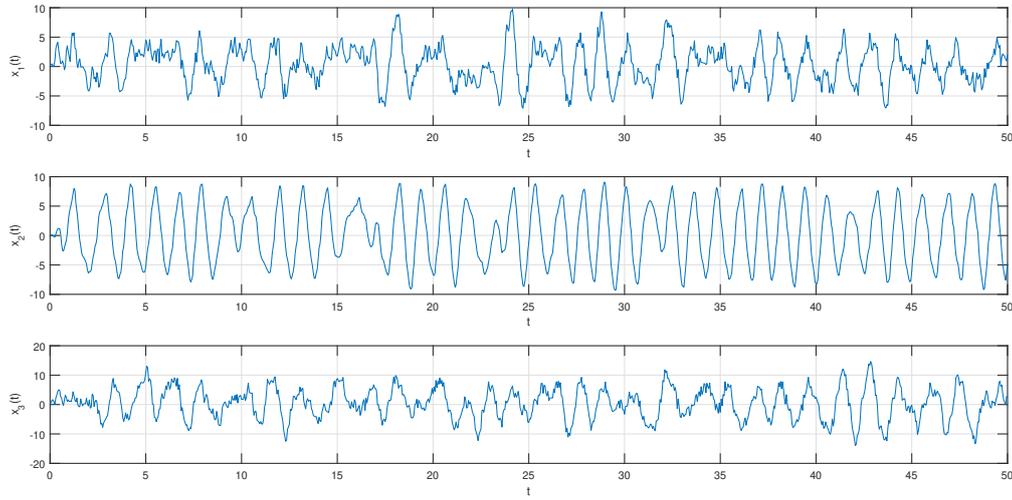


Figure 2. Chaotic behavior of the incommensurate fractional-order neural networks (12)

then the roots λ , and their appropriate arguments of polynomial (13) are

$$\begin{aligned} \lambda_1 &= 11.0064 + 0.0000i, & |arg(\lambda_1)| &= 0 \\ \lambda_2 &= -0.8000 + 0.0000i, & |arg(\lambda_2)| &= 3.1416 \\ \lambda_{3,4} &= 0.0968 + 0.8120i, & |arg(\lambda_4)| &= 1.4521 \end{aligned}$$

x^* is unstable for the given fractional-orders because the eigenvalue λ_1 is a positive real integer.

system (12) is considered the master system. Now, Let's consider the slave system

$$\begin{cases} {}_0^C D_t^{\alpha_1} y_1(t) = -c_1 y_1(t) + b_{11} g(x_1(t)) + b_{12} g(y_2(t)) + b_{13} g(y_3(t)) + I_1 + U_1 \\ {}_0^C D_t^{\alpha_2} y_2(t) = -c_2 y_2(t) + b_{21} g(y_1(t)) + b_{22} g(y_2(t)) + b_{23} g(y_3(t)) + I_2 + U_2 \\ {}_0^C D_t^{\alpha_3} y_3(t) = -c_3 y_3(t) + b_{31} g(y_1(t)) + b_{32} g(y_2(t)) + b_{33} g(y_3(t)) + I_3 + U_3 \end{cases} \quad (14)$$

Theorem 3.2 states that there exists a control matrix A such that C and A fulfills the condition. One can, for example, select the form

$$A = \text{diag} \left(\begin{bmatrix} 1.75 & 1.6 & 0.7 \end{bmatrix} \right)$$

This obviously meets the condition, as a result, systems (12) and (14) are synchronized. It is now quite

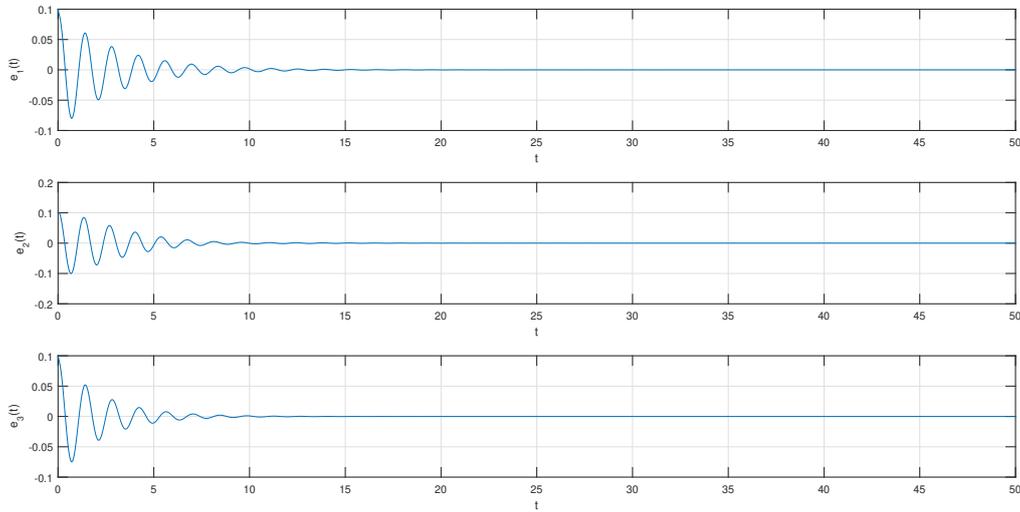


Figure 3. The evolution of the error system's states

simple to create the control law using Theorem 3.2. The final error system looks such as this

$$\begin{cases} {}^C_0 D_t^{\alpha_1} e_1(t) = -0.95e_1(t) \\ {}^C_0 D_t^{\alpha_2} e_2(t) = -0.8e_2(t) \\ {}^C_0 D_t^{\alpha_3} e_3(t) = -0.9e_3(t) \end{cases} \quad (15)$$

Figure 3 depicts the temporal development of the errors. Clearly, synchronization is accomplished when the errors converge to zero in a reasonable amount of time.

5. Conclusion

The stability of incommensurate fractional-order neural networks is investigated in this research. Several interesting stability criteria are obtained from the introduced characteristic equation. We successfully derive a suitable stability requirement for incommensurate fractional-order neural networks using the data obtained. We also extend our conclusions to the synchronization of chaotic incommensurate fractional-order neural networks. Two numerical examples have been provided to illustrate the effectiveness of the theoretical analysis.

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